Statistics of velocity fluctuations arising from a random distribution of point vortices: The speed of fluctuations and the diffusion coefficient

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This paper is devoted to a statistical analysis of the fluctuations of velocity and acceleration produced by a random distribution of point vortices in two-dimensional turbulence. We show that the velocity probability density function PDF behaves in a manner which is intermediate between Gaussian and Lévy laws, while the distribution of accelerations is governed by a Cauchy law. Our study accounts properly for a spectrum of circulations among the vortices. In the case of real vortices (with a finite core), we show analytically that the distribution of accelerations makes a smooth transition from Cauchy (for small fluctuations) to Gaussian (for large fluctuations), probably passing through an exponential tail. We introduce a function $T(V)$ which gives the typical duration of a velocity fluctuation V; we show that $T(V)$ behaves like V and V^{-1} for weak and large velocities, respectively. These results have a simple physical interpretation in the nearest neighbor approximation, and in Smoluchowski theory concerning the persistence of fluctuations. We discuss the analogies with respect to the fluctuations of the gravitational field in stellar systems. As an application of these results, we determine an approximate expression for the diffusion coefficient of point vortices. When applied to the context of freely decaying two-dimensional turbulence, the diffusion becomes anomalous and we establish a relationship $\nu=1+(\xi/2)$ between the exponent of anomalous diffusion ν and the exponent ξ which characterizes the decay of the vortex density.

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I. INTRODUCTION

A basic problem in fluid turbulence is the characterization of the entire stochastic variation of the velocity field $\mathbf{V}(\mathbf{r},t)$ produced by the disordered motion of the flow. The velocity fluctuations can be described by different quantities such as their probability density function, the typical duration of the fluctuations, and their spatial or temporal correlations. We consider a simple model of two-dimensional turbulence for which it is possible to calculate these quantities exactly. In our model, the velocity is produced by a collection of point vortices randomly distributed in the domain with uniform probability. Point vortices behave like particles in interaction, and share some common features with electric charges [1] or stars [2–5]. In particular, the problem at hand is directly connected with the problem of the fluctuations of the electric field in a gas composed of simple ions or the fluctuations of the gravitational field produced by a random distribution of stars. These problems were considered by Holtsmark [6] in electrostatics and Chandrasekhar [7] and Chandrasekhar and von Neumann [8,9] in a stellar context. We will show that many of the methods introduced by these authors can be extended to the case of point vortices, even if the calculations, and the results differ due to the lower dimensionality of space ($D=2$ instead of $D=3$) and the different nature of the interactions.

We consider a collection of N point vortices randomly distributed in a disk of radius R. We assume that the vortices have a Poisson distribution, i.e., their positions are independent and uniformly distributed over the entire domain. We are particularly interested in the ‘‘thermodynamical limit’’ in which the number of vortices and the size of the domain go to infinity ($N\to\infty, R\to\infty$) in such a way that the vortex density $n=N/\pi R^2$ remains finite. In this limit, the Poisson distribution is shown to be stationary (see, e.g., Ref. [10]) and is well suited to the analysis of the fluctuations. For the moment, the vortices have the same circulation γ, but we shall indicate later how the results can be generalized to include a spectrum of circulations.

The velocity $\mathbf{V}$ occurring at a given location of the flow is the sum of the velocities $\mathbf{\Phi}_i$ ($i=1,\dots,N$) produced by the N vortices:

$$\mathbf{V}=\sum_{i=1}^{N}\mathbf{\Phi}_i, \qquad (1)$$

$$\mathbf{\Phi}_i=-\frac{\gamma}{2\pi}\frac{\mathbf{r}_{\perp i}}{r_i^2}, \qquad (2)$$

where $\mathbf{r}_i$ denotes the position of the ith vortex relative to the point under consideration and, by definition, $\mathbf{r}_{\perp i}$ is the vector $\mathbf{r}_i$ rotated by $+\pi/2$. Since the vortices are randomly distributed, the velocity $\mathbf{V}$ fluctuates. It is therefore of interest to study the statistics of these fluctuations, i.e., the probability $W(\mathbf{V})d^2\mathbf{V}$ that $\mathbf{V}$ lies between $\mathbf{V}$ and $\mathbf{V}+d\mathbf{V}$. We find that this distribution behaves in a manner which is intermediate between Gaussian and Lévy laws: the core of the distribution is Gaussian, with ‘‘variance’’

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$$\langle V^2\rangle = \frac{n\gamma^2}{4\pi}\ln N, \tag{3}$$

while the high velocity tail decreases algebraically like V^{-4}. Since the ''variance'' behaves like $\sim \ln N$, the thermodynamical limit is not well defined and the results are polluted by logarithmic corrections. Previous investigations of this problem were carried out numerically in Refs. [11–13].

However, we must be aware that the knowledge of $W(\mathbf{V})$ alone does not provide us with all the necessary information concerning fluctuations of $\mathbf{V}$. An important aspect of the problem concerns the *speed of fluctuations*, i.e., the typical duration $T(V)$ of the velocity fluctuation $\mathbf{V}$. This requires the knowledge of the bivariate probability $W(\mathbf{V},\mathbf{A})d^2\mathbf{V}d^2\mathbf{A}$ to measure simultaneously a velocity $\mathbf{V}$ with a rate of change

$$\mathbf{A} = \frac{d\mathbf{V}}{dt} = \sum_{i=1}^{N} \boldsymbol{\psi}_i, \tag{4}$$

$$\boldsymbol{\psi}_i = -\frac{\gamma}{2\pi}\left(\frac{\mathbf{v}_{\perp i}}{r_i^2} - \frac{2(\mathbf{r}_i\cdot\mathbf{v}_i)\mathbf{r}_{\perp i}}{r_i^4}\right), \tag{5}$$

where $\mathbf{v}_i = d\mathbf{r}_i/dt$ is the velocity of vortex i. Then the duration $T(V)$ can be estimated by the formula

$$T(V) = \frac{|\mathbf{V}|}{\sqrt{\langle A^2\rangle_{\mathbf{V}}}}, \tag{6}$$

where

$$\langle A^2\rangle_{\mathbf{V}} = \frac{\int W(\mathbf{V},\mathbf{A})A^2 d^2\mathbf{A}}{W(\mathbf{V})} \tag{7}$$

is the mean square acceleration associated with a velocity fluctuation $\mathbf{V}$. A similar quantity was introduced in Refs. [8,9] in a stellar context. We find that the distribution of the accelerations is governed by a Cauchy law, and that the typical duration $T(V)$ of a velocity fluctuation $\mathbf{V}$ behaves like V for $V\to 0$ and V^{-1} for $V\to\infty$. We also establish that the average duration of the velocity fluctuations is

$$\langle T\rangle \sim \frac{1}{n\gamma\sqrt{\ln N}}. \tag{8}$$

These results can be understood in the ''nearest neighbor approximation,'' in which the most proximate vortex plays a prevalent role. In this approximation, the determination of the speed of fluctuations can be deduced from the theory of Smoluchowski [14] concerning the mean lifetime of a state with X particles.

In terms of the previous quantities, we can estimate the diffusion coefficient of point vortices by the formula

$$D = \frac{1}{4}\int T(\mathbf{V})W(\mathbf{V})V^2 d^2\mathbf{V}. \tag{9}$$

We find that

$$D \sim \gamma\sqrt{\ln N}, \tag{10}$$

and we discuss qualitatively how the formation of ''pairs'' modifies the results of our study. In the context of freely decaying two-dimensional turbulence, the diffusion coefficient is time dependent (since the circulation of a vortex increases as a result of successive mergings), and the diffusion is anomalous. From Eq. (10), we establish a relationship

$$\nu = 1 + \frac{\xi}{2} \tag{11}$$

between the exponent of anomalous diffusion ν and the exponent ξ which characterizes the decay of the vortex density. This relation is in good agreement with laboratory experiments (Hansen *et al.* [15]) and numerical simulations (Sire and Chavanis [16]).

We indicate how our results are modified when we allow for a spectrum of circulations among the vortices. This is an important generalization, since decaying two-dimensional (2D) turbulence possesses a continuous distribution of vortices. We show that the distribution of velocity and acceleration are only slightly modified by the polydispersity of the vortices, and we justify the validity of previous comparisons of full numerical simulations with vortex models that ignored this difference (e.g., Ref. [11]).

Finally, we generalize our results to the case of vortex ''blobs'' with a finite core. We show that the natural cutoff at $r=a$, the vortex radius, removes the algebraic tail of the velocity distribution. Further, we analytically show that the distribution of accelerations makes a smooth transition from Cauchy (for small fluctuations) to Gaussian (for large fluctuations). It is likely that in between the distribution passes through an *exponential tail* as observed numerically in Ref. [12] for the velocity gradients.

II. STATISTICS OF VELOCITY FLUCTUATIONS

A. General expression for $W(\mathbf{V})$

We shall now obtain the distribution $W_N(\mathbf{V})$ of the velocity $\mathbf{V}$ produced by N point vortices randomly distributed in a disk of radius R with uniform probability. To avoid a possible solid rotation, we shall assume that the system is ''neutral,'' the circulation of the vortices taking only two values $+\gamma$ and $-\gamma$ in equal proportion. Since a vortex with circulation $-\gamma$ located in $\mathbf{r}$ produces the same velocity as a vortex with circulation $+\gamma$ located in $-\mathbf{r}$, and since the vortices are randomly distributed over the entire domain with uniform probability, the group of vortices with negative circulation is statistically equivalent to the group of vortices with positive circulation. We can therefore proceed as if there were a single species of particles but no solid rotation. Since we shall ultimately let $R\to\infty$, we can assume without loss of generality that $\mathbf{V}$ is calculated at the center of the domain.

Under these circumstances, the distribution $W_N(\mathbf{V})$ can be expressed as

$$W_N(\mathbf{V}) = \int \prod_{i=1}^{N} \tau(\mathbf{r}_i)d^2\mathbf{r}_i\,\delta\left(\mathbf{V} - \sum_{i=1}^{N}\boldsymbol{\Phi}_i\right), \tag{12}$$

where $\tau(\mathbf{r}_i)d^2\mathbf{r}_i$ governs the probability of occurrence of the ith point vortex at position $\mathbf{r}_i$. In writing this expression, we have assumed that the vortices are identical and uncorrelated.

Now, using a method originally due to Markov, we express the δ function appearing in Eq. (12) in terms of its Fourier transform:

$$\delta(\mathbf{x}) = \frac{1}{(2\pi)^2} \int e^{-i\boldsymbol{\rho}\cdot\mathbf{x}} d^2\boldsymbol{\rho}. \tag{13}$$

With this transformation, $W_N(\mathbf{V})$ becomes

$$W_N(\mathbf{V}) = \frac{1}{4\pi^2} \int A_N(\boldsymbol{\rho}) e^{-i\boldsymbol{\rho}\cdot\mathbf{V}} d^2\boldsymbol{\rho}, \tag{14}$$

with

$$A_N(\boldsymbol{\rho}) = \left(\int_{|\mathbf{r}|=0}^{R} e^{i\boldsymbol{\rho}\cdot\boldsymbol{\Phi}} \tau(\mathbf{r}) d^2\mathbf{r} \right)^N, \tag{15}$$

where we have written

$$\boldsymbol{\Phi} = -\frac{\gamma}{2\pi} \frac{\mathbf{r}_\perp}{r^2}. \tag{16}$$

If we now suppose that the vortices are uniformly distributed on average, then

$$\tau(\mathbf{r}) = \frac{1}{\pi R^2}, \tag{17}$$

and Eq. (15) reduces to

$$A_N(\boldsymbol{\rho}) = \left(\frac{1}{\pi R^2} \int_{|\mathbf{r}|=0}^{R} e^{i\boldsymbol{\rho}\cdot\boldsymbol{\Phi}} d^2\mathbf{r} \right)^N. \tag{18}$$

Since

$$\frac{1}{\pi R^2} \int_{|\mathbf{r}|=0}^{R} d^2\mathbf{r} = 1, \tag{19}$$

we can rewrite our expression for $A_N(\boldsymbol{\rho})$ in the form

$$A_N(\boldsymbol{\rho}) = \left(1 - \frac{1}{\pi R^2} \int_{|\mathbf{r}|=0}^{R} (1 - e^{i\boldsymbol{\rho}\cdot\boldsymbol{\Phi}}) d^2\mathbf{r} \right)^N. \tag{20}$$

We now consider the limit when the number of vortices and the size of the domain go to infinity in such a way that the density remains finite:

$$N \to \infty, \quad R \to \infty, \quad n = \frac{N}{\pi R^2} \quad \text{finite}.$$

If the integral occurring in Eq. (20) increases less rapidly than N, then

$$A(\boldsymbol{\rho}) = e^{-nC(\boldsymbol{\rho})}, \tag{21}$$

with

$$C(\boldsymbol{\rho}) = \int_{|\mathbf{r}|=0}^{R} (1 - e^{i\boldsymbol{\rho}\cdot\boldsymbol{\Phi}}) d^2\mathbf{r}. \tag{22}$$

We have dropped the subscript N to indicate that the limit $N \to \infty$, in the previous sense, has been taken. Note that $A(\boldsymbol{\rho})$ can still depend on N through logarithmic factors, so that Eq. (21) must be considered as an equivalent of Eq. (20) for large N's, not a true limit.

To calculate $C(\boldsymbol{\rho})$ explicitly, it is more convenient to introduce $\boldsymbol{\Phi}$ as a variable of integration instead of $\mathbf{r}$. The Jacobian of the transformation $\{\mathbf{r}\} \to \{\boldsymbol{\Phi}\}$ is

$$\left\| \frac{\partial(\mathbf{r})}{\partial(\boldsymbol{\Phi})} \right\| = \frac{\gamma^2}{4\pi^2 \Phi^4}, \tag{23}$$

so that

$$C(\boldsymbol{\rho}) = \frac{\gamma^2}{4\pi^2} \int_{|\boldsymbol{\Phi}|=\gamma/2\pi R}^{+\infty} (1 - e^{i\boldsymbol{\rho}\cdot\boldsymbol{\Phi}}) \frac{1}{\Phi^4} d^2\boldsymbol{\Phi}, \tag{24}$$

or, alternatively,

$$C(\boldsymbol{\rho}) = \frac{\gamma^2}{4\pi^2} \int_{|\boldsymbol{\Phi}|=\gamma/2\pi R}^{+\infty} [1 - \cos(\boldsymbol{\rho}\cdot\boldsymbol{\Phi})] \frac{1}{\Phi^4} d^2\boldsymbol{\Phi}. \tag{25}$$

Choosing polar coordinates with the x axis in the direction of $\boldsymbol{\rho}$, Eq. (25) can be transformed to

$$C(\boldsymbol{\rho}) = \frac{\gamma^2}{4\pi^2} \int_{\gamma/2\pi R}^{+\infty} \frac{d\Phi}{\Phi^3} \int_0^{2\pi} [1 - \cos(\rho\Phi\cos\theta)] d\theta, \tag{26}$$

where θ denotes the angle between $\boldsymbol{\rho}$ and $\boldsymbol{\Phi}$. Using the identity

$$\int_0^{\pi} \cos(z\cos\theta) d\theta = \pi J_0(z), \tag{27}$$

we obtain

$$C(\boldsymbol{\rho}) = \frac{\gamma^2}{2\pi} \int_{\gamma/2\pi R}^{+\infty} [1 - J_0(\rho\Phi)] \frac{d\Phi}{\Phi^3}, \tag{28}$$

or, writing $x = \rho\Phi$,

$$C(\boldsymbol{\rho}) = \frac{\gamma^2\rho^2}{2\pi} \int_{\gamma\rho/2\pi R}^{+\infty} [1 - J_0(x)] \frac{dx}{x^3}. \tag{29}$$

Recall that $C(\boldsymbol{\rho})$ must be evaluated in the limit $N, R \to \infty$, with $n = N/\pi R^2$ finite. Using the well-known expansion of the Bessel function J_0 for small arguments,

$$J_0(x) = 1 - \frac{x^2}{4} + o(x^4), \tag{30}$$

we have the estimate

$$C(\boldsymbol{\rho}) = \frac{\gamma^2\rho^2}{16\pi} \ln\left(\frac{4\pi N}{n\gamma^2\rho^2} \right). \tag{31}$$

Since $C(\boldsymbol{\rho})$ diverges weakly with N (logarithmically), the limiting process leading to formula (21) is permissible. For $\rho > 0$ and $N \to \infty$, we have

$$A(\boldsymbol{\rho})=e^{-(n\gamma^2/16\pi)\ln N\rho^2}, \quad (32)$$

and for $\rho\to 0$, we obtain

$$A(\boldsymbol{\rho})=e^{(n\gamma^2/8\pi)(\ln\rho)\rho^2}. \quad (33)$$

The velocity distribution $W(\mathbf{V})$ is simply the Fourier transform of $A(\boldsymbol{\rho})$. We shall now derive the expression for $W(\mathbf{V})$ in the core and in the tail of the distribution.

B. Core of the distribution $W(\mathbf{V})$

For $V\lesssim V_{crit}(N)$, where $V_{crit}(N)$ is defined by formula (50), the contribution of small ρ's in integral (14) is negligible, and we can use expression (32) for $A(\boldsymbol{\rho})$. In that case, the distribution $W(\mathbf{V})$ is the Gaussian

$$W(\mathbf{V})=\frac{4}{n\gamma^2\ln N}e^{-(4\pi/n\gamma^2\ln N)V^2} \quad [V\lesssim V_{crit}(N)]. \quad (34)$$

If we were to extend this distribution for all values of V, we would conclude that its variance

$$\langle V^2\rangle=\frac{n\gamma^2}{4\pi}\ln N \quad (35)$$

diverges logarithmically when $N\to\infty$. This result was noted by Jiménez [11], Min *et al.* [12], and Weiss *et al.* [13], who applied a generalized form of the central limit theorem. In fact, the central limit theorem is not strictly applicable here, because the variance of the velocity created by a single vortex,

$$\langle\Phi^2\rangle=\int_{|\mathbf{r}|=0}^{R}\tau(\mathbf{r})\Phi^2 d^2\mathbf{r}=\int_0^R\frac{1}{\pi R^2}\frac{\gamma^2}{4\pi^2 r^2}2\pi r\,dr, \quad (36)$$

diverges logarithmically; still, the distribution of $\mathbf{V}$ is Gaussian [for $V\lesssim V_{crit}(N)$], but its "variance" behaves like $\ln N$. For $V\gtrsim V_{crit}(N)$, distribution (34) breaks down because, for large velocities, the Fourier transform (14) is dominated by the contribution of small ρ's, and formula (33) must be used instead of Eq. (32). This implies that the high velocity tail of the distribution $W(\mathbf{V})$ decays algebraically like V^{-4} (see Sec. II C). This algebraic tail arises because we are on the frontier between Gaussian and Lévy laws (see Fig. 1.1 of Ref. [17], and Sec. V).

Distribution (34) has been derived for a neutral system consisting in an equal number of vortices with circulation $+\gamma$ and $-\gamma$. In Appendix B, we extend our results to allow for a spectrum of circulations among the vortices, still for a neutral system. If the system is non-neutral, there is a solid rotation and the average velocity increases linearly with the distance. Therefore, at point $\mathbf{a}$, Eq. (34) must be replaced by

$$W(\mathbf{V})=\frac{4}{n\gamma^2\ln N}e^{-(4\pi/n\gamma^2\ln N)[\mathbf{V}-(1/2)n\gamma\mathbf{a}_\perp]^2}$$
$$\left(\left|\mathbf{V}-\frac{1}{2}n\gamma\mathbf{a}_\perp\right|\lesssim V_{crit}(N)\right). \quad (37)$$

The velocity distribution at $\mathbf{a}\neq\mathbf{0}$ differs only from the distribution at the center of the domain by replacing the velocity $\mathbf{V}$ by the fluctuating velocity $\boldsymbol{\mathcal{V}}=\mathbf{V}-\langle\mathbf{V}\rangle=\mathbf{V}-\frac{1}{2}n\gamma\mathbf{a}_\perp$. A factor 1/2 arises in front of the average vorticity $n\gamma$ because, for a solid rotation, the vorticity is twice the angular velocity.

C. High velocity tail of the distribution $W(\mathbf{V})$

We shall now determine the behavior of the distribution $W(\mathbf{V})$ for $V\to\infty$. Introducing polar coordinates with the x axis in the direction of $\mathbf{V}$, and using Eq. (21), Eq. (14) can be transformed to

$$W(\mathbf{V})=\frac{1}{2\pi^2}\int_0^{\pi}d\theta\int_0^{+\infty}e^{-i\rho V\cos\theta}e^{-nC(\rho)}\rho\,d\rho. \quad (38)$$

With the change of variables $z=\rho V$ and $t=-\cos\theta$, Eq. (38) can be rewritten

$$W(\mathbf{V})=\frac{1}{2\pi^2V^2}\mathrm{Re}_e\int_{-1}^{+1}\frac{dt}{\sqrt{1-t^2}}\int_0^{+\infty}e^{izt}e^{-nC(z/V)}z\,dz. \quad (39)$$

In this expression, t and z are real, and the domains of integration $\tau_0: -1\leqslant t\leqslant 1$ and $\zeta_0: 0\leqslant z<+\infty$ are taken along the real axis. Under these circumstances, the integral is not convergent if we expand the quantity $\exp(-nC(z/V))$ in a power series of z/V, for $V\to+\infty$, and evaluate the integral term by term. However, regarding z and t as complex variables, it is possible to choose paths of integration along which this expansion will converge.

We shall first carry out the integration on z, for a fixed t. It will therefore be possible to choose the (complex) integration paths for z dependent on t. The integration paths are modified as follows: τ_0 is replaced by τ, the semicircle with radius unity lying in the domain $\mathrm{Im}(t)\geqslant 0$. Therefore, $\arg(t)$ varies from π to 0 when t moves from -1 to $+1$. On the other hand, ζ_0 is replaced by ζ_{ω_t}, the line starting from the origin and forming an angle

$$\omega_t=\frac{1}{8}\left(\frac{\pi}{2}-\arg(t)\right) \quad (40)$$

with the real axis. When t moves from -1 to $+1$, ω_t varies from $-\pi/16$ to $+\pi/16$. For $|z|\to\infty$, according to Eq. (32), we have

$$e^{-nC(z/V)}=e^{-(n\gamma^2/16\pi)\ln N(z^2/V^2)}. \quad (41)$$

Since the argument of z^2 is between $-\pi/8$ and $\pi/8$, its real part is always positive, and the convergence of Eq. (39) is undisturbed. On the other hand, the argument of izt is equal to $\pi/2+1/8[\pi/2-\arg(t)]+\arg(t)$, and lies between $9\pi/16$ and $23\pi/16$. Therefore, the real part of izt is always negative, and the function e^{izt} decays exponentially to zero as

$|z|\to\infty$. Therefore, with the new paths of integration τ and ζ_{ω_t}, it is possible to expand the integrand of Eq. (39) in power series of z/V, for $V\to\infty$, and integrate term by term. When $z/V\to 0$, we have, according to Eq. (33),

$$e^{-nC(z/V)}=e^{(n\gamma^2/8\pi)\ln(z/V)(z^2/V^2)}, \tag{42}$$

and we can write

$$W(\mathbf{V})=\frac{1}{2\pi^2V^2}\mathrm{Re}\int_\tau\frac{dt}{\sqrt{1-t^2}}\int_{\zeta_{\omega_t}}e^{izt}\times\left[1+\frac{n\gamma^2}{8\pi}\ln\left(\frac{z}{V}\right)\frac{z^2}{V^2}+\cdots\right]zdz. \tag{43}$$

Since this integral is convergent along any line on which the real part of izt is negative, we can replace the integration path ζ_{ω_t} by the line ζ_{ψ_t}, forming an angle

$$\psi_t=\frac{\pi}{2}-\arg(t) \tag{44}$$

with the real axis. On this new integration path $izt=-y$ (y real$\geqslant 0$), and we obtain

$$W(\mathbf{V})=-\frac{1}{2\pi^2V^2}\mathrm{Re}\int_{-1}^{+1}\frac{dt}{\sqrt{1-t^2}}\int_0^{+\infty}e^{-y}\times\left[1-\frac{n\gamma^2}{8\pi}\ln\left(\frac{iy}{V}\right)\frac{1}{t^2}\frac{y^2}{V^2}+\frac{n\gamma^2}{8\pi}\frac{\ln t}{t^2}\frac{y^2}{V^2}+\cdots\right]\frac{y}{t^2}dy, \tag{45}$$

where we recall that t is a complex variable, and the integration has to be performed over the semicircle of radius unity lying on the domain $\mathrm{Im}(t)\geqslant 0$. Writing $t=e^{i\theta}$, we find that

$$\int_{-1}^{+1}\frac{dt}{t^2\sqrt{1-t^2}}=0,\quad \int_{-1}^{+1}\frac{dt}{t^4\sqrt{1-t^2}}=0,$$

$$\int_{-1}^{+1}\frac{\ln t}{t^4\sqrt{1-t^2}}dt=-\frac{2\pi}{3}. \tag{46}$$

Therefore,

$$W(\mathbf{V})=\frac{n\gamma^2}{24\pi^2V^4}\int_0^{+\infty}e^{-y}y^3dy. \tag{47}$$

In this expression, we recognize the Γ function

$$\Gamma(n+1)=\int_0^{+\infty}e^{-y}y^n dy, \tag{48}$$

with $n=3$. Its value is $\Gamma(4)=6$, and we finally obtain

$$W(\mathbf{V})=\frac{n\gamma^2}{4\pi^2V^4}\quad [V\gtrsim V_{crit}(N)]. \tag{49}$$

Therefore, for sufficiently large values of V, the velocity distribution $W(\mathbf{V})$ decays algebraically, like V^{-4}. In Sec. V, we give a physical interpretation of this result in terms of the nearest neighbor approximation.

From Eqs. (34) and (49), we can estimate the value of the velocity $V_{crit}(N)$ for which the distribution $W(\mathbf{V})$ departs from the Gaussian. $V_{crit}(N)$ is obtained by seeking the point where the two regimes (34) and (49) connect to each other. Neglecting subdominant terms in $\ln N$, one simply finds

$$V_{crit}(N)\sim\left(\frac{n\gamma^2}{4\pi}\ln N\right)^{1/2}\ln^{1/2}(\ln N). \tag{50}$$

This result shows that the convergence to a pure Gaussian distribution is extremely slow, with N as emphasized in Refs. [11–13]. Since the distribution $W(\mathbf{V})$ decreases like V^{-4} for $V\to\infty$, the variance of the velocity diverges logarithmically.

Note, finally, that the distribution of V_x, the x component of the velocity, is

$$W(V_x)=\frac{2}{\sqrt{n\gamma^2\ln N}}e^{-4\pi V_x^2/n\gamma^2\ln N}\quad [V_x\lesssim V_{crit}(N)], \tag{51}$$

$$W(V_x)=\frac{n\gamma^2}{8\pi V_x^3}\quad [V_x\gtrsim V_{crit}(N)]. \tag{52}$$

D. Formation of ‘‘pairs’’

The previous results should be all the more valid if the velocity $\mathbf{V}$ is calculated at a fixed point of the domain. In such a case, there is no restriction on the possible values of V since a vortex can approach this point with no limit producing extremely large velocities. The situation is different if $\mathbf{V}$ is now the velocity experienced by a ‘‘test’’ vortex. Indeed, if a ‘‘field’’ vortex approaches the test vortex below a certain distance, then a ‘‘pair’’ will form, and our treatment, which ignores the correlations between vortices, will clearly break down. These pairs have been observed and studied numerically by Weiss *et al.* [13].

We can simply estimate the typical separation below which a pair will form by comparing the velocity produced by a single vortex $\gamma/2\pi r$ with the typical velocity $V_{typ}=[(n\gamma^2/4\pi)\ln N]^{1/2}$ produced by the field [see Eq. 35]. This yields

$$d_{pair}(N)=(\pi n\ln N)^{-1/2}, \tag{53}$$

a distance slightly smaller than the interparticle distance by a factor $\sim 1/\sqrt{\ln N}$. In the mathematical limit $N\to\infty$, there is no pair, since $d_{pair}\to 0$. This result is in agreement with the stationarity of the Poisson process when $N\to\infty$: if the vortices are initially uncorrelated and uniformly distributed, they will remain uncorrelated and uniformly distributed. However, the convergence is extremely slow with N, and close pairs will always form in realistic situations. As emphasized by Weiss *et al.* [13], a system of 10^3–10^5 vortices has a behavior which is a combination of both low-dimensional behavior, i.e., closed pairs, and high-dimensional behavior described by traditional stochastic processes.

A pair can be either a ''binary'' (rapidly rotating around its center of vorticity), when two vortices of the same sign are bound together, or a ''dipole'' (translating or rotating), when two vortices of opposite sign pair off. Of course, binaries and dipoles behave very differently. If the vortex is engaged in a binary with a long correlation time, then, for practical purposes, the relevant velocity to consider is not its own velocity (which has a rotating component), but rather the velocity of the center of vorticity which is induced by the rest of the system. Therefore, a binary simply behaves like a single point vortex with a larger circulation and a relatively slow velocity (otherwise this means that the binary is itself engaged in a pair). It may be noted that, in the case of real vortices (with a finite core), the formation of binaries is replaced by merging events. On the other hand, a dipole moves by itself and behaves like a kind of particle undergoing fast ballistic motion. Its velocity may be large but, since it creates a dipolar velocity field, the previous results cannot be applied directly, and an appropriate treatment is required.

We therefore expect that the velocity distributions (34) and (49), which ignore correlations between vortices, will break down for $V \gg V_{crit}(N)$ since, in that case, the velocity is entirely due to the nearest neighbor, and pairs form. In the following, we shall account for this failure by introducing a cutoff at some V_{max}, i.e.,

$$W(\mathbf{V})=0 \quad (V>V_{max}). \tag{54}$$

This is the simplest modification that we can make to account for the formation of pairs at large velocities. Likewise, in the stellar context, the formation of binaries alters the results of the stochastic analysis at large field strengths.

III. STATISTICS OF ACCELERATIONS

A. General formula for $W(\mathbf{V},\mathbf{A})$

Here we are concerned with a calculation of the bivariate probability $W_N(\mathbf{V},\mathbf{A})$ to measure simultaneously a velocity $\mathbf{V}$ with a rate of change $\mathbf{A}=d\mathbf{V}/dt$. According to Eqs. (1) and (4), $\mathbf{V}$ and $\mathbf{A}$ are the sum of N random variables $\mathbf{\Phi}_i$ and $\boldsymbol{\psi}_i$ depending on the positions $\mathbf{r}_i$ and velocities $\mathbf{v}_i$ of the point vortices. However, unlike material particles, the variables $\{\mathbf{r}_i,\mathbf{v}_i\}$, for different i's, are not independent because the velocities of the vortices are determined by the configuration $\{\mathbf{r}_i\}$ of the system as a whole. However, for our purpose, it is probably a reasonable approximation to neglect these correlations and treat $\{\mathbf{r}_i,\mathbf{v}_i\}$ $(i=1,\ldots,N)$ as independent variables. We shall only describe qualitatively how the formation of pairs affects our results.

When this decorrelation hypothesis is implemented, a straightforward generalization of the method used in Sec. II A yields

$$W_N(\mathbf{V},\mathbf{A})=\frac{1}{16\pi^4}\int A_N(\boldsymbol{\rho},\boldsymbol{\sigma})e^{-i(\boldsymbol{\rho}\cdot\mathbf{V}+\boldsymbol{\sigma}\cdot\mathbf{A})}d^2\boldsymbol{\rho}\,d^2\boldsymbol{\sigma}, \tag{55}$$

with

$$A_N(\boldsymbol{\rho},\boldsymbol{\sigma})=\left(\int_{|\mathbf{r}|=0}^{R}\int_{|\mathbf{v}|=0}^{+\infty}\tau(\mathbf{r},\mathbf{v})e^{i(\boldsymbol{\rho}\cdot\mathbf{\Phi}+\boldsymbol{\sigma}\cdot\boldsymbol{\psi})}d^2\mathbf{r}\,d^2\mathbf{v}\right)^N, \tag{56}$$

where we have defined

$$\mathbf{\Phi}=-\frac{\gamma}{2\pi}\frac{\mathbf{r}_\perp}{r^2}, \tag{57}$$

$$\boldsymbol{\psi}=-\frac{\gamma}{2\pi}\left(\frac{\mathbf{v}_\perp}{r^2}-\frac{2(\mathbf{r}\cdot\mathbf{v})\mathbf{r}_\perp}{r^4}\right), \tag{58}$$

and where $\tau(\mathbf{r},\mathbf{v})$ denotes the probability that a vortex be in $\mathbf{r}$ with velocity $\mathbf{v}$. According to our initial assumptions, the vortices are distributed uniformly on average, and $\tau(\mathbf{r})$ is given by Eq. (17). On the other hand, their velocity distribution $\tau(\mathbf{v})$ is given by Eqs. (34) and (49) of Sec. II. However, due to the formation of pairs, this distribution must be modified at large velocities (see Sec. II D). Instead of introducing a sharp cutoff $\tau(\mathbf{v})=0$ at $v>v_{max}$, we shall assume for convenience that the Gaussian distribution (34) is valid for all velocities. Therefore, the probability that a vortex be in $\mathbf{r}$ with velocity $\mathbf{v}$ is

$$\tau(\mathbf{r},\mathbf{v})=\frac{1}{\pi R^2}\frac{4}{n\gamma^2\ln N}e^{-(4\pi/n\gamma^2\ln N)v^2}q. \tag{59}$$

It is remarkable that distribution (59) is formally equivalent to the Maxwell-Boltzmann statistics of material particles at equilibrium. Owing to this analogy, we can interpret the variance

$$\overline{v^2}=\frac{n\gamma^2}{4\pi}\ln N \tag{60}$$

as a kind of kinetic ''temperature.'' More generally, the moment of order p of the velocity is

$$\overline{v^p}=\left(\frac{n\gamma^2\ln N}{4\pi}\right)^{p/2}\Gamma\left(\frac{p}{2}+1\right), \tag{61}$$

where the Γ function is defined by Eq. (48). In particular

$$\overline{v}=\left(\frac{n\gamma^2}{16}\ln N\right)^{1/2}. \tag{62}$$

Recall that the distribution (59) is valid only for a neutral system made of an equal number of vortices with circulation $+\gamma$ and $-\gamma$ (if the system is non-neutral, we must account for a solid rotation). Since a vortex with circulation $-\gamma$, located in $\mathbf{r}$ and moving with velocity $\mathbf{v}$, produces the same velocity $\mathbf{V}$ *and* acceleration $\mathbf{A}$ as a vortex with circulation $+\gamma$ located in $-\mathbf{r}$ and moving with velocity $-\mathbf{v}$, and since the vortices are randomly distributed with a uniform probability and isotropic velocity distribution, the two groups of vortices are statistically equivalent. Therefore, as in Sec. II, we can proceed as if we had a single type of vortex with circulation γ and no solid rotation. In Appendix B, we extend our results to allow for a spectrum of circulations among the vortices and an arbitrary isotropic distribution of the velocity $\tau(|\mathbf{v}|)$.

Substituting expression (59) for $\tau(\mathbf{r},\mathbf{v})$ into Eq. (56), we obtain

$$A_N(\boldsymbol{\rho},\boldsymbol{\sigma})=\left(\frac{1}{\pi R^2}\int_{|\mathbf{r}|=0}^{R}\int_{|\mathbf{v}|=0}^{+\infty}\frac{4}{n\gamma^2\ln N}\times e^{-(4\pi/n\gamma^2\ln N)v^2}e^{i(\boldsymbol{\rho}\cdot\boldsymbol{\Phi}+\boldsymbol{\sigma}\cdot\boldsymbol{\psi})}d^2\mathbf{r}d^2\mathbf{v}\right)^N. \tag{63}$$

As in Sec. II A, it is more convenient to use $\boldsymbol{\Phi}$ and $\boldsymbol{\psi}$ as variables of integration rather than $\mathbf{r}$ and $\mathbf{v}$. The Jacobian of the transformation $\{\mathbf{r},\mathbf{v}\}\rightarrow\{\boldsymbol{\Phi},\boldsymbol{\psi}\}$ is

$$\left\|\frac{\partial(\mathbf{r},\mathbf{v})}{\partial(\boldsymbol{\Phi},\boldsymbol{\psi})}\right\|=\frac{\gamma^4}{16\pi^4\Phi^8}. \tag{64}$$

We must next express $v=|\mathbf{v}|$ in terms of our new variables $\boldsymbol{\Phi}$ and $\boldsymbol{\psi}$. According to Eqs. (57) and (58), we have $\Phi=\gamma/2\pi r$ and $\psi=\gamma v/2\pi r^2$. Hence

$$v=\frac{\gamma}{2\pi}\frac{\psi}{\Phi^2}. \tag{65}$$

Thus, in these new variables, the expression for $A_N(\boldsymbol{\rho},\boldsymbol{\sigma})$ becomes

$$A_N(\boldsymbol{\rho},\boldsymbol{\sigma})=\left(\frac{1}{\pi R^2}\int_{|\boldsymbol{\Phi}|=\gamma/2\pi R}^{+\infty}\int_{|\boldsymbol{\psi}|=0}^{+\infty}\frac{4}{n\gamma^2\ln N}\times e^{-(1/n\pi\ln N)(\psi^2/\Phi^4)}e^{i(\boldsymbol{\rho}\cdot\boldsymbol{\Phi}+\boldsymbol{\sigma}\cdot\boldsymbol{\psi})}\times\frac{\gamma^4}{16\pi^4\Phi^8}d^2\boldsymbol{\Phi}d^2\boldsymbol{\psi}\right)^N. \tag{66}$$

The integral with respect to $\boldsymbol{\psi}$ is Gaussian and is readily evaluated. We are left with

$$A_N(\boldsymbol{\rho},\boldsymbol{\sigma})=\left(\frac{1}{\pi R^2}\int_{|\boldsymbol{\Phi}|=\gamma/2\pi R}^{+\infty}e^{i\boldsymbol{\rho}\cdot\boldsymbol{\Phi}}e^{-\pi n\ln N\sigma^2\Phi^4/4}\times\frac{\gamma^2}{4\pi^2\Phi^4}d^2\boldsymbol{\Phi}\right)^N. \tag{67}$$

We verify that

$$\frac{1}{\pi R^2}\int_{|\boldsymbol{\Phi}|=\gamma/2\pi R}^{+\infty}\frac{\gamma^2}{4\pi^2\Phi^4}d^2\boldsymbol{\Phi}=1. \tag{68}$$

Hence, we can rewrite Eq. (67) in the form

$$A_N(\boldsymbol{\rho},\boldsymbol{\sigma})=\left(1-\frac{1}{\pi R^2}\int_{|\boldsymbol{\Phi}|=\gamma/2\pi R}^{+\infty}(1-e^{i\boldsymbol{\rho}\cdot\boldsymbol{\Phi}}\times e^{-\pi n\ln N\sigma^2\Phi^4/4})\frac{\gamma^2}{4\pi^2\Phi^4}d^2\boldsymbol{\Phi}\right)^N. \tag{69}$$

We shall now consider the limit $N,R\rightarrow\infty$ with $n=N/\pi R^2$ finite. If the integral appearing in Eq. (69) increases less rapidly than N, then

$$A(\boldsymbol{\rho},\boldsymbol{\sigma})=e^{-nC(\boldsymbol{\rho},\boldsymbol{\sigma})}, \tag{70}$$

with

$$C(\boldsymbol{\rho},\boldsymbol{\sigma})=\frac{\gamma^2}{4\pi^2}\int_{|\boldsymbol{\Phi}|=\gamma/2\pi R}^{+\infty}(1-e^{i\boldsymbol{\rho}\cdot\boldsymbol{\Phi}}e^{-\pi n\ln N\sigma^2\Phi^4/4})\frac{1}{\Phi^4}d^2\boldsymbol{\Phi}. \tag{71}$$

As in Sec. II A, the function $A(\boldsymbol{\rho},\boldsymbol{\sigma})$ represents an equivalent of $A_N(\boldsymbol{\rho},\boldsymbol{\sigma})$ for large N, not a true limit. It can therefore still depend on N through logarithmic factors.

After introducing polar coordinates and integrating over the angular variable using Eq. (27), we arrive at

$$C(\boldsymbol{\rho},\boldsymbol{\sigma})=\frac{\gamma^2}{2\pi}\int_{\gamma/2\pi R}^{+\infty}(1-J_0(\rho\Phi)e^{-\pi n\ln N\sigma^2\Phi^4/4})\frac{d\Phi}{\Phi^3}. \tag{72}$$

Equations (55), (70), and (72) formally solve the problem, but the integrals look difficult to calculate explicitly. However, if we are only interested in the moments of $\mathbf{A}$ for a given $\mathbf{V}$ (or in the moments of $\mathbf{V}$ for a given $\mathbf{A}$), we only need the asymptotic expansion of $A(\boldsymbol{\rho},\boldsymbol{\sigma})$ for $|\boldsymbol{\sigma}|\rightarrow 0$ ($|\boldsymbol{\rho}|\rightarrow 0$). This problem will be considered in Sec. III C. First we shall derive the unconditional distribution of the acceleration.

B. Cauchy distribution for A

According to Eq. (55), we clearly have

$$W(\mathbf{A})=\frac{1}{16\pi^4}\int A(\boldsymbol{\rho},\boldsymbol{\sigma})e^{-i(\boldsymbol{\rho}\cdot\mathbf{V}+\boldsymbol{\sigma}\cdot\mathbf{A})}d^2\boldsymbol{\rho}d^2\boldsymbol{\sigma}d^2\mathbf{V}. \tag{73}$$

Using Eq. (13), the foregoing expression for $W(\mathbf{A})$ reduces to

$$W(\mathbf{A})=\frac{1}{4\pi^2}\int A(\boldsymbol{\sigma})e^{-i\boldsymbol{\sigma}\cdot\mathbf{A}}d^2\boldsymbol{\sigma}, \tag{74}$$

where we have written $A(\boldsymbol{\sigma})$ for $A(\mathbf{0},\boldsymbol{\sigma})$. Hence, according to Eqs. (70) and (72), we obtain

$$A(\boldsymbol{\sigma})=e^{-nC(\boldsymbol{\sigma})}, \tag{75}$$

with

$$C(\boldsymbol{\sigma})=\frac{\gamma^2}{2\pi}\int_0^{+\infty}(1-e^{-\pi n\ln N\sigma^2\Phi^4/4})\frac{d\Phi}{\Phi^3}. \tag{76}$$

Following the usual prescription, we have let $R\to\infty$, since the integral is convergent when $\Phi\to 0$. Integrating by parts, we obtain

$$C(\boldsymbol{\sigma})=\frac{\gamma^2}{8}\sqrt{n\ln N}\sigma. \tag{77}$$

Hence

$$A(\boldsymbol{\sigma})=e^{-\gamma^2 n^{3/2}/8\sqrt{\ln N}\sigma}. \tag{78}$$

The distribution $W(\mathbf{A})$ is simply the Fourier transform of the exponential function (78). This is the 2D Cauchy distribution:

$$W(\mathbf{A})=\frac{32}{\pi n^3\gamma^4\ln N}\frac{1}{\left(1+\frac{64A^2}{n^3\gamma^4\ln N}\right)^{3/2}}. \tag{79}$$

The Cauchy distribution is a particular Lévy law. As such, it decays algebraically for large $|\mathbf{A}|$'s, according to

$$W(\mathbf{A})\sim\frac{\gamma^2 n^{3/2}\sqrt{\ln N}}{16\pi A^3}\quad (A\to\infty). \tag{80}$$

This result has a clear physical interpretation in the nearest neighbor approximation (see Sec. V). Only the moments of order $p<1$ of the acceleration exist, and we have the general expression

$$\langle A^p\rangle=\left(\frac{n^{3/2}\gamma^2\sqrt{\ln N}}{8}\right)^p\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1-p}{2}\right)\Gamma\left(1+\frac{p}{2}\right). \tag{81}$$

Note, finally, that the distribution of A_x, the x component of the acceleration, is the ordinary 1D Cauchy law

$$W(A_x)=\frac{8}{\pi n^{3/2}\gamma^2\sqrt{\ln N}}\frac{1}{1+\frac{64A_x^2}{n^3\gamma^4\ln N}}. \tag{82}$$

We can show, furthermore, that the distribution of accelerations is related to the distribution of velocity gradients $\delta\mathbf{V}$ (in preparation). This is to be expected since $\mathbf{A}=d\mathbf{V}/dt$.

C. Moments $\langle A^2\rangle_{\mathbf{V}}$ and $\langle V^2\rangle_{\mathbf{A}}$

Let us introduce a Cartesian system of coordinates, and denote by $\{A_\mu\}$ the different components of $\mathbf{A}$ relative to that frame. The average value of $A_\mu A_\nu$ for a given velocity $\mathbf{V}$ is defined by

$$\langle A_\mu A_\nu\rangle_{\mathbf{V}}=\frac{1}{W(\mathbf{V})}\int W(\mathbf{V},\mathbf{A})A_\mu A_\nu d^2\mathbf{A}. \tag{83}$$

According to Eq. (55), it can be written

$$W(\mathbf{V})\langle A_\mu A_\nu\rangle_{\mathbf{V}}=\frac{1}{16\pi^4}\int A(\boldsymbol{\rho},\boldsymbol{\sigma})\times e^{-i(\boldsymbol{\rho}\cdot\mathbf{V}+\boldsymbol{\sigma}\cdot\mathbf{A})}A_\mu A_\nu d^2\boldsymbol{\rho}d^2\boldsymbol{\sigma}d^2\mathbf{A}, \tag{84}$$

or, equivalently,

$$W(\mathbf{V})\langle A_\mu A_\nu\rangle_{\mathbf{V}}=-\frac{1}{16\pi^4}\int A(\boldsymbol{\rho},\boldsymbol{\sigma})\frac{\partial^2}{\partial\sigma^\mu\partial\sigma^\nu}\times\{e^{-i(\boldsymbol{\rho}\cdot\mathbf{V}+\boldsymbol{\sigma}\cdot\mathbf{A})}\}d^2\boldsymbol{\rho}d^2\boldsymbol{\sigma}d^2\mathbf{A}. \tag{85}$$

Integrating by parts, we obtain

$$W(\mathbf{V})\langle A_\mu A_\nu\rangle_{\mathbf{V}}=-\frac{1}{16\pi^4}\int\frac{\partial^2 A}{\partial\sigma^\mu\partial\sigma^\nu}(\boldsymbol{\rho},\boldsymbol{\sigma})\times e^{-i(\boldsymbol{\rho}\cdot\mathbf{V}+\boldsymbol{\sigma}\cdot\mathbf{A})}d^2\boldsymbol{\rho}d^2\boldsymbol{\sigma}d^2\mathbf{A}. \tag{86}$$

Using identity (13), we can readily carry out the integration on $\mathbf{A}$ and $\boldsymbol{\sigma}$, to finally obtain

$$W(\mathbf{V})\langle A_\mu A_\nu\rangle_{\mathbf{V}}=-\frac{1}{4\pi^2}\int\frac{\partial^2 A}{\partial\sigma^\mu\partial\sigma^\nu}(\boldsymbol{\rho},\mathbf{0})e^{-i\boldsymbol{\rho}\cdot\mathbf{V}}d^2\boldsymbol{\rho}. \tag{87}$$

Since the characteristic function $A(\boldsymbol{\rho},\boldsymbol{\sigma})$ is isotropic in each of its variables [see Eqs. (70) and (72)], the tensor $\langle A_\mu A_\nu\rangle_{\mathbf{V}}$ is diagonal, and can be expressed as

$$\langle A_\mu A_\nu\rangle_{\mathbf{V}}=\frac{1}{2}\langle A^2\rangle_{\mathbf{V}}\delta_{\mu\nu}, \tag{88}$$

where $\langle A^2\rangle_{\mathbf{V}}$ is given by

$$W(\mathbf{V})\langle A^2\rangle_{\mathbf{V}}=-\frac{1}{\pi^2}\int\frac{\partial A}{\partial(\sigma^2)}(\boldsymbol{\rho},\mathbf{0})e^{-i\boldsymbol{\rho}\cdot\mathbf{V}}d^2\boldsymbol{\rho}. \tag{89}$$

We therefore need the behavior of $A(\boldsymbol{\rho},\boldsymbol{\sigma})$ for $|\boldsymbol{\sigma}|\to 0$, or according to Eq. (70) the behavior of $C(\boldsymbol{\rho},\boldsymbol{\sigma})$ for $|\boldsymbol{\sigma}|\to 0$. Introducing the function $C(\boldsymbol{\rho})$ defined in Sec. II A [see Eq. (28)], and using Eq. (72), we can write

$$C(\boldsymbol{\rho},\boldsymbol{\sigma})=C(\boldsymbol{\rho})+F(\boldsymbol{\rho},\boldsymbol{\sigma}), \tag{90}$$

with

$$F(\boldsymbol{\rho},\boldsymbol{\sigma})=\frac{\gamma^2}{2\pi}\int_0^{+\infty}J_0(\rho\Phi)(1-e^{-\pi n\ln N\sigma^2\Phi^4/4})\frac{d\Phi}{\Phi^3}. \tag{91}$$

We have let $R\to\infty$, since the integral is convergent when $\Phi\to 0$. In terms of this new function, the expression for $\langle A^2\rangle_{\mathbf{V}}$ can be rewritten

$$W(\mathbf{V})\langle A^2\rangle_{\mathbf{V}}=\frac{n}{\pi^2}\int e^{-nC(\boldsymbol{\rho})}\frac{\partial F}{\partial(\sigma^2)}(\boldsymbol{\rho},\mathbf{0})e^{-i\boldsymbol{\rho}\cdot\mathbf{V}}d^2\boldsymbol{\rho}. \tag{92}$$

It is shown in Appendix A that

$$\frac{\partial F}{\partial(\sigma^2)}(\boldsymbol{\rho},\mathbf{0})=\frac{\pi n}{4}\gamma^2\ln N\,\delta(\boldsymbol{\rho}), \tag{93}$$

where δ stands for the Dirac δ function. Substituting the foregoing expression in Eq. (92), we obtain

$$W(\mathbf{V})\langle A^2\rangle_{\mathbf{V}}=\frac{n^2\gamma^2}{4\pi}\ln N. \tag{94}$$

Therefore, $\langle A^2\rangle_{\mathbf{V}}$ behaves like the inverse of the velocity distribution $W(\mathbf{V})$. Combining Eqs. (94), (34), and (49), we obtain

$$\langle A^2\rangle_{\mathbf{V}}=\frac{n^3\gamma^4}{16\pi}(\ln N)^2 e^{(4\pi V^2/n\gamma^2\ln N)} \quad [V\lesssim V_{crit}(N)], \tag{95}$$

$$\langle A^2\rangle_{\mathbf{V}}=n\pi\ln N V^4 \quad [V\gtrsim V_{crit}(N)]. \tag{96}$$

For $V\to\infty$, $\langle A^2\rangle_{\mathbf{V}}$ behaves like V^4. This result finds a simple physical interpretation in the nearest neighbor approximation (see Sec. V).

By a similar procedure, we can calculate the variance of the velocity for an assigned rate of change. We find

$$\langle V^2\rangle_{\mathbf{A}}=\frac{n\gamma^2}{4\pi}\ln N \quad [A\lesssim A_{crit}(N)], \tag{97}$$

$$\langle V^2\rangle_{\mathbf{A}}=\frac{2A}{\pi\sqrt{n\ln N}} \quad [A\gtrsim A_{crit}(N)]. \tag{98}$$

For moderate values of A, the variance $\langle V^2\rangle_{\mathbf{A}}$ is independent of A, and coincides with formula (35). For large fluctuations, result (98) can be recovered in the nearest neighbor approximation (see Sec. V). The crossover between the two distributions (97) and (98) occurs at

$$A_{crit}(N)\sim\frac{n^{3/2}\gamma^2}{8}(\ln N)^{3/2}. \tag{99}$$

IV. SPEED OF FLUCTUATIONS AND THE DIFFUSION COEFFICIENT

A. Mean lifetime of a velocity fluctuation V

On the basis of very general considerations (see, e.g., Ref. [13]), we would expect that the typical duration of the velocity fluctuations be

$$T_{typ}\sim\frac{d}{\sqrt{\langle V^2\rangle}}. \tag{100}$$

This corresponds to the time needed by a vortex with typical velocity $\sqrt{\langle V^2\rangle}$ to cross the interparticle distance $d\sim n^{-1/2}$. Using expression (35) for $\langle V^2\rangle$, we find

$$T_{typ}\sim\frac{1}{n\gamma\sqrt{\ln N}}. \tag{101}$$

Apart from the logarithmic correction, this formula can be obtained by direct dimensional analysis. However, if we define the mean lifetime of a state V by the formula

$$T(V)=\frac{|\mathbf{V}|}{\sqrt{\langle A^2\rangle_{\mathbf{V}}}}, \tag{102}$$

the theory developed along the previous lines enables us to obtain a more precise characterization of the speed of fluctuations depending on their intensity. This definition is consistent with the definition introduced in Refs. [8,9] in the context of stellar dynamics. Of course, Eq. (102) is just an order of magnitude, but it should reasonably well account for the dependence on the speed of fluctuations with V. Using Eqs. (95) and (96), we find, explicitly,

$$T(V)=\frac{4\sqrt{\pi}V}{n^{3/2}\gamma^2\ln N}e^{-(2\pi/n\gamma^2\ln N)V^2} \quad [V\lesssim V_{crit}(N)], \tag{103}$$

$$T(V)=\frac{1}{\sqrt{\pi n\ln N}}\frac{1}{V} \quad [V\gtrsim V_{crit}(N)]. \tag{104}$$

For weak and large fluctuations, $T(V)$ decreases to zero like V and V^{-1}, respectively. These asymptotic behaviors are consistent with the theory developed by Smoluchowski [14] in his general investigation on fluctuation phenomena (see Sec. V C). These results (and those of Sec. III) should be all the more valid if $\mathbf{V}$ is calculated at a fixed point. By contrast, if $\mathbf{V}$ denotes the velocity experienced by a test vortex, we expect some discrepancies at large V's due to the formation of pairs. In such a case, the correlation time can be extremely long (in particular for binaries).

The average duration of the fluctuations is defined by

$$\langle T\rangle=\int_0^{+\infty}T(V)W(V)2\pi V dV. \tag{105}$$

To leading order in $\ln N$ [i.e., extending Eqs. (34) and (103) to all V's], we obtain

$$\langle T\rangle=\frac{4}{3}\left(\frac{\pi}{6}\right)^{1/2}\frac{1}{n\gamma\sqrt{\ln N}}, \tag{106}$$

in good agreement with estimate (101) based on general physical grounds.

B. Diffusion coefficient

According to the previous discussion, we can characterize the fluctuations of the velocity of a point vortex (or a passive particle) by two functions: a function $W(\mathbf{V})$ which governs the occurrence of the velocity $\mathbf{V}$, and a function $T(V)$ which determines the typical time during which the vortex moves with this velocity. Since the velocity fluctuates on a typical time $T_{typ}=d/\sqrt{\langle V^2\rangle}$, which is much smaller than the dynamical time $T_D=R/\sqrt{\langle V^2\rangle}$ needed by the vortex to cross the entire domain, the motion of the vortex will be essentially stochastic. If we denote by $P(\mathbf{r},t)$ the probability density that the particle be found in $\mathbf{r}$ at time t, then $P(\mathbf{r},t)$ will satisfy the diffusion equation

$$\frac{\partial P}{\partial t}=D\Delta P. \tag{107}$$

If the particle is at $\mathbf{r}=\mathbf{r}_0$ at time $t=0$, the solution of Eq. (107) is clearly

$$P(\mathbf{r},t|\mathbf{r}_0)=\frac{1}{4\pi Dt}e^{-(|\mathbf{r}-\mathbf{r}_0|^2)/4Dt}, \tag{108}$$

where D is the diffusion coefficient. The mean square displacement that the particle is expected to suffer during an interval of time Δt, large with respect to the fluctuation time T_{typ}, is

$$\langle(\Delta\mathbf{r})^2\rangle=4D\Delta t. \tag{109}$$

We can obtain another expression for $\langle(\Delta\mathbf{r})^2\rangle$ in terms of the functions $W(\mathbf{V})$ and $T(V)$ defined in the previous sections. Indeed, dividing the interval

$$\Delta\mathbf{r}=\int_t^{t+\Delta t}\mathbf{V}(t')dt' \tag{110}$$

into a succession of discrete increments in position with amount $T(V_i)\mathbf{V}_i$, we readily establish that

$$\langle(\Delta\mathbf{r})^2\rangle=\langle T(V)V^2\rangle\Delta t. \tag{111}$$

Combining Eqs. (109) and (111) we obtain an alternative expression for the diffusion coefficient in the form

$$D=\frac{1}{4}\int T(V)W(\mathbf{V})V^2d^2\mathbf{V}. \tag{112}$$

Substituting for $T(V)$ and $W(\mathbf{V})$ in the foregoing expression, we obtain, to leading order in $\ln N$,

$$D=\frac{1}{72}\left(\frac{6}{\pi}\right)^{1/2}\gamma\sqrt{\ln N}. \tag{113}$$

We should not give too much credit to the numerical factor appearing in Eq. (113), since definition (102) of $T(V)$ is just an order of magnitude. Note that the functional form of D is consistent with the expression

$$D\sim T_{typ}\langle V^2\rangle\sim\gamma\sqrt{\ln N} \tag{114}$$

that one would expect on general physical grounds. Weiss *et al.* [13] proposed describing the motion of the vortices by a Ornstein-Uhlenbeck process with ''friction'' $-\mathbf{V}/\langle T\rangle$ to take into account the finite decorrelation time of the system. The function $T(V)$ introduced in the present paper could be used instead of $\langle T\rangle$, to take into account the dependence of the decorrelation time with the strength V of the fluctuations.

The present theory ignores the formation of pairs since we have formally introduced a cutoff at large V's. This cutoff is justified for binaries since, as we have already explained, the relevant velocity to consider is the velocity of the center of vorticity, not the velocity of the individual vortices engaged in the pair. This is not the case for dipoles which can make relatively long jumps from one point to another with an almost ballistic motion. Weiss *et al.* [13] proposed interpreting these jumps in terms of Lévy walks responsible for anomalous diffusion. Therefore, the present theory should provide reliable results only for relatively short times. For times $t\gg T_D$, the pairs must be taken into account and anomalous diffusion will ensue. The case of passive particles is not so different. A passive particle is advected by the other vortices but has no influence on their motion. However, passives can become trapped in the vicinity of a vortex undergoing fast dipolar motion, and also experience Lévy walks.

The previous results remain valid for a non-neutral system rotating uniformly, provided that the velocity $\mathbf{V}$ is replaced by the fluctuating velocity $\boldsymbol{\mathcal{V}}=\mathbf{V}-\langle\mathbf{V}\rangle$. In particular, expressions (106) and (113) for $\langle T\rangle$ and D are unchanged. For a differential rotation, the fluctuation time is related to the local shear Σ, as investigated by Chavanis [3] using an approximation in which the point vortices are simply transported by the mean flow. Therefore, the present theory gives the value of the fluctuation time in regions where the shear cancels out. Of course, a more general calculation should take into account the effect of both the shear and the dispersion of particles, but this will not be considered here. Note also that when the system is inhomogeneous, the point vortices are subjected to a *systematic drift* [3] in addition to their diffusive motion. This drift may be responsible for the organization of point vortices at ''negative temperatures'' [18].

C. Application to 2D decaying turbulence

Let us consider a collection of vortices of size a, vorticity ω, and density n. Due to merging events, their size increases with time as the density decreases. The typical core vorticity ω remains constant during the course of the evolution as suggested in Refs. [19–21], and observed experimentally in Ref. [15]. These authors found that the density decreases as $n\sim t^{-\xi}$ with $\xi\approx0.7$. As the energy $E\sim N\omega^2a^4$ is conserved throughout the merging process, the typical vortex radius is $a\sim t^{\xi/4}$. Since the average distance between vortices, of order $d\sim t^{\xi/2}$ increases more rapidly than their size, the point vortex model should provide increasing accuracy. We can therefore expect that the vortices will diffuse with a coefficient $D\sim\gamma$ [see Eq. (113)], where $\gamma\sim\omega a^2$ is their circulation (we ignore logarithmic corrections). If the diffusion coefficient were constant, then the dispersion of the vortices

$$\langle r^2\rangle\sim Dt \tag{115}$$

would increase linearly with time, as in ordinary Brownian motion. However, since D varies with time according to

$$D\sim\omega a^2\sim t^{\xi/2}, \tag{116}$$

we expect anomalous diffusion, i.e.,

$$\langle r^2\rangle\sim t^{\nu}, \tag{117}$$

with $\nu\neq1$. Substituting Eq. (116) into Eq. (115), we obtain the following relation between ν and ξ:

$$\nu=1+\frac{\xi}{2}. \tag{118}$$

This expression differs from formula (19) of Ref. [15] because their estimate of D is different. These authors estimate the diffusion coefficient by $D\sim\tau_{merg}\langle V^2\rangle$, where τ_{merg} is the average time between two successive mergings of a given vortex. By a simple cross section argument, they obtained $\tau_{merg}\sim 1/(n\sqrt{\langle V^2\rangle}a)$, which is larger than $\tau_{fluct}\sim d/\sqrt{\langle V^2\rangle}$ by a factor d/a. This shows that the merging time is not the relevant correlation time to consider in the diffusion process. In fact, using $\xi\approx 0.7$, formula (118) leads to $\nu\approx 1.35$, in better agreement with the experimental value $\nu\approx 1.3$ ($\nu\approx 1.4$ for passive particles) than their relation $\nu=1+(3\xi/4)$, leading to $\nu\approx 1.53$. Formula (118) is also in perfect agreement with the numerical simulations of Ref. [16].

V. NEAREST-NEIGHBOR APPROXIMATION

A. Importance of the nearest neighbor

The velocity $\mathbf{V}$ and acceleration $\mathbf{A}$ experienced by a test vortex (or occurring at a fixed point) are the sum of N random variables $\mathbf{\Phi}_i$ and $\boldsymbol{\psi}_i$ produced by all the vortices present in the system [see Eqs. (1) and (4)]. In each sum, the highest term is due to the nearest neighbor, at an average distance $d\sim n^{-1/2}$ from the point under consideration. This single vortex creates a typical velocity and acceleration:

$$V_{\rm NN}^2\sim\left(\frac{\gamma}{2\pi d}\right)^2\sim\frac{\gamma^2}{4\pi^2}\frac{N}{\pi R^2}, \tag{119}$$

$$A_{\rm NN}^2\sim\overline{v^2}\left(\frac{\gamma}{2\pi d^2}\right)^2\sim\frac{\gamma^2}{4\pi^2}\overline{v^2}\left(\frac{N}{\pi R^2}\right)^2. \tag{120}$$

It is interesting to compare the contribution of the nearest neighbor with the contribution of the other $N-1$ vortices. For that purpose, we estimate the typical value of V and A produced by *all* the vortices by

$$V^2\sim N\left\langle\frac{\gamma^2}{4\pi^2r^2}\right\rangle\sim N\int_{|\mathbf{r}|=d}^{R}\tau(\mathbf{r})\frac{\gamma^2}{4\pi^2r^2}d^2\mathbf{r}\sim\frac{\gamma^2}{4\pi}\frac{N}{\pi R^2}\ln N, \tag{121}$$

$$A^2\sim N\overline{v^2}\left\langle\frac{\gamma^2}{4\pi^2r^4}\right\rangle\sim N\overline{v^2}\int_{|\mathbf{r}|=d}^{R}\tau(\mathbf{r})\frac{\gamma^2}{4\pi^2r^4}d^2\mathbf{r}\sim\frac{\gamma^2}{4\pi}\overline{v^2}\left(\frac{N}{\pi R^2}\right)^2. \tag{122}$$

If the variance of $\mathbf{\Phi}$ and $\boldsymbol{\psi}$ were finite, the central limit theorem would be applicable, and the variables V and A would scale like $\sqrt{N}$. In that case, none of the terms in sums (1) and (4) would have a dominant contribution, and the scaling $\sqrt{N}$ would simply reflect the *collective* behavior of the system. This is not the case, however, in the present situation, since the variance of $\mathbf{\Phi}$ and $\boldsymbol{\psi}$ diverge. The variance of $\boldsymbol{\psi}$ diverges algebraically, and, thus, the acceleration produced by all the vortices is dominated by the contribution of the nearest neighbor. This *individual* nature is a specific and striking property of a Lévy law. For a Lévy law, the sum of N random variables behaves like the largest term [compare Eqs. (122) and (120)]. The case of the velocity is particular because the variance of $\mathbf{\Phi}$ diverges only logarithmically. First considering Eq. (121), and neglecting the logarithmic correction, we observe that the velocity produced by all the vortices behaves like $\sqrt{N}$, as though the central limit theorem were applicable. However, comparing with Eq. (119), we note that $\sqrt{N}$ is also the scaling of the largest term in the sum. Therefore, the contribution of the nearest neighbor is of the same order as the contribution of the rest of the system (up to a logarithmic factor). It is on account of this particular circumstance that the velocity has a behavior which is intermediate between Gaussian and Lévy laws, as remarked upon earlier. In a sense, we can consider that the velocity is dominated by the contribution of the nearest neighbor, and that collective effects are responsible for logarithmic corrections.

B. Distribution due to the nearest neighbor

In light of the previous discussion, it is interesting to analyze in more detail the distribution of the velocity and acceleration produced by the nearest neighbor. For that purpose, we must first determine the probability $\tau_{\rm NN}(r)dr$ that the position of the nearest neighbor occurs between r and $r+dr$. Clearly, $\tau_{\rm NN}(r)dr$ is equal to the probability that no vortices exist interior to r times the probability that a vortex (any) exists in the annulus between r and $r+dr$. Therefore, it must satisfy an equation of the form

$$\tau_{\rm NN}(r)dr=\left(1-\int_0^r\tau_{\rm NN}(r')dr'\right)n2\pi r dr, \tag{123}$$

where $n=N/\pi R^2$ denotes the mean density of vortices in the disk. Differentiating with respect to r, we obtain

$$\frac{d}{dr}\left[\frac{\tau_{\rm NN}(r)}{2\pi n r}\right]=-\tau_{\rm NN}(r). \tag{124}$$

This equation is readily integrated with the condition $\tau_{\rm NN}(r)\sim 2\pi n r$ as $r\to 0$, and we find

$$\tau_{\rm NN}(r)=2\pi n r e^{-\pi n r^2}. \tag{125}$$

This is the distribution of the nearest neighbor in a random distribution of particles. From this formula, we can obtain the exact value for the ''average distance'' d between vortices. By definition,

$$d=\int_0^{+\infty}\tau_{\rm NN}(r)r dr. \tag{126}$$

Hence

$$d=\frac{1}{2\sqrt{n}}. \tag{127}$$

The probability of finding the nearest neighbor in $\mathbf{r}$ with velocity $\mathbf{v}$ is

$$\tau_{\rm NN}(\mathbf{r},\mathbf{v})=ne^{-\pi n r^2}\frac{4}{n\gamma^2\ln N}e^{-(4\pi/n\gamma^2\ln N)v^2}. \tag{128}$$

If we assume that the velocity **V** and the acceleration **A** are entirely due to the nearest neighbor, then

$$W_{NN}(\mathbf{V},\mathbf{A})d^2\mathbf{V}d^2\mathbf{A}=\tau_{NN}(\mathbf{r},\mathbf{v})d^2\mathbf{r}d^2\mathbf{v}, \tag{129}$$

with

$$\mathbf{V}=-\frac{\gamma}{2\pi}\frac{\mathbf{r}_\perp}{r^2} \tag{130}$$

$$\mathbf{A}=-\frac{\gamma}{2\pi}\left(\frac{\mathbf{v}_\perp}{r^2}-\frac{2(\mathbf{r}\cdot\mathbf{v})\mathbf{r}_\perp}{r^4}\right). \tag{131}$$

Since the Jacobian of the transformation $\{\mathbf{r},\mathbf{v}\}\to\{\mathbf{V},\mathbf{A}\}$ is

$$\left\|\frac{\partial(\mathbf{r},\mathbf{v})}{\partial(\mathbf{V},\mathbf{A})}\right\|=\frac{\gamma^4}{16\pi^4V^8}, \tag{132}$$

we obtain

$$W_{NN}(\mathbf{V},\mathbf{A})=\frac{\gamma^2}{4\pi^4\ln N}\frac{1}{V^8}e^{-(n\gamma^2/4\pi V^2)}e^{-(A^2/n\pi\ln NV^4)}, \tag{133}$$

where we have used $r=\gamma/2\pi V$ and $v=\gamma A/2\pi V^2$. The nearest neighbor approximation is expected to give relevant results only for large values of the velocity and the acceleration. Thus we can make the additional approximation

$$W_{NN}(\mathbf{V},\mathbf{A})=\frac{\gamma^2}{4\pi^4\ln N}\frac{1}{V^8}e^{-(A^2/n\pi\ln NV^4)}. \tag{134}$$

Integrating on the acceleration, we find

$$W_{NN}(\mathbf{V})=\frac{n\gamma^2}{4\pi^2V^4}, \tag{135}$$

in perfect agreement with Eq. (49) valid for $V\gtrsim V_{crit}$. This shows that the algebraic tail of the velocity distribution is produced by the nearest neighbor. This is characteristic of a Lévy law. On the other hand, for $V\lesssim V_{crit}$, the velocity distribution is Gaussian, as if the central limit theorem were applicable. Once again, the simultaneous occurrence of collective and individual behaviors is a manifestation of the very peculiar nature of an interaction in r^{-1} in two dimensions.

Integrating on the velocity, we find

$$W_{NN}(\mathbf{A})=\frac{\gamma^2n^{3/2}\sqrt{\ln N}}{16\pi A^3}, \tag{136}$$

in perfect agreement with the asymptotic behavior of the Cauchy distribution [Eq. (80)]. We also establish that

$$\langle A^2\rangle_{\mathbf{V}}=n\pi\ln NV^4, \tag{137}$$

$$\langle V^2\rangle_{\mathbf{A}}=\frac{2A}{\pi\sqrt{n\ln N}}, \tag{138}$$

in complete agreement with formulas (96) and (98).

C. Application of Smoluchowski theory

In the nearest neighbor approximation, the duration of the velocity fluctuations can be deduced from the theory of Smoluchowski [14] concerning the persistence of fluctuations. This approach was used by Chandrasekhar [7] in his elementary analysis of the fluctuations of the gravitational field. An account of Smoluchowski theory can be found in Ref. [22]. In the case of point vortices, it leads to the formula

$$T(V)=\frac{\gamma}{4\overline{v}}\frac{V}{\frac{n\gamma^2}{4\pi}+V^2}, \tag{139}$$

where $V=\gamma/2\pi r$ is the velocity due to the most proximate vortex, at a distance r from the point under consideration. The Smoluchowski formula [Eq. (139)] has the same asymptotic behaviors as Eqs. (103) and (104). These asymptotic behaviors have a clear physical meaning. When $r=\gamma/2\pi V$ is small, corresponding to large velocities, it is highly improbable that another vortex will enter the disk of radius r before long. By contrast, on a short time scale $T\sim r/\overline{v}\sim\gamma/\overline{v}V$, the vortex will have left the disk. When $r=\gamma/2\pi V$ is large, corresponding to small velocities, the probability that the vortex will remain alone in the disk is low. The characteristic time before another vortex enters the disk varies as the inverse of the number of vortices expected to be present in the disk, i.e., $T\sim(r/\overline{v})(1/n\pi r^2)\sim V/n\gamma\overline{v}$. The demarcation between weak and strong fluctuations corresponds to $V\sim\gamma n^{1/2}$, i.e. to the velocity produced by a vortex distant $n^{-1/2}$ from the point under consideration.

VI. CONCLUSION

In this paper, we have analyzed in detail the statistics of velocity fluctuations produced by a random distribution of point vortices. We have determined the velocity distribution and the speed of fluctuations. We have also shown how some of the results can be understood in the nearest neighbor approximation. Our results should be accurate if the velocity is calculated at a fixed point. However, there should be substantial discrepancies at large velocities if **V** now represents the velocity experienced by a point vortex. This is due to the formation of pairs (binaries or dipoles) when two vortices come into contact. If we ignore these pairs, the motion of the vortices is purely diffusive, and we determined the functional form of the diffusion coefficient. In the case of real vortices, with a finite core, the formation of binaries is replaced by merging events, and the number of vortices decreases with time. This results in anomalous diffusion. We proposed a relationship between the exponent of anomalous diffusion ν and the exponent ξ which characterizes the decay of the vortex density. In a future study (in preparation), we shall be concerned with the spatial correlations of the velocity.

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Note added. Recently we have become aware of the work by Kuvshinov and Schep (to appear in Phys. Rev. Lett.) on the statistics of point vortex systems.

APPENDIX A: DERIVATION OF FORMULA (93)

By definition,

$$F(\boldsymbol{\rho},\boldsymbol{\sigma})=\frac{\gamma^2}{2\pi}\int_0^{+\infty}J_0(\rho\Phi)(1-e^{-\pi n\ln N\sigma^2\Phi^4/4})\frac{d\Phi}{\Phi^3}. \tag{A1}$$

For $\boldsymbol{\rho}=\mathbf{0}$, we have already found [see Eqs. (76) and (77)] that

$$F(\mathbf{0},\boldsymbol{\sigma})=\frac{\gamma^2}{8}\sqrt{n\ln N}\sigma. \tag{A2}$$

Therefore,

$$\frac{\partial F}{\partial(\sigma^2)}(\boldsymbol{\rho},\mathbf{0})=+\infty \quad \text{if} \quad \boldsymbol{\rho}=\mathbf{0}. \tag{A3}$$

For $\boldsymbol{\rho}\neq\mathbf{0}$, we can make the change of variable $z=\rho\Phi$ in Eq. (A1). This yields

$$F(\boldsymbol{\rho},\boldsymbol{\sigma})=\frac{\gamma^2\rho^2}{2\pi}\int_0^{+\infty}J_0(z)(1-e^{-\pi n\ln N(\sigma^2/\rho^4)z^4/4})\frac{dz}{z^3}. \tag{A4}$$

We therefore have to determine the behavior of the function

$$f(p)=\int_0^{+\infty}J_0(z)(1-e^{-pz^4})\frac{dz}{z^3}, \tag{A5}$$

as $p\rightarrow 0$. Clearly, it is not possible to expand the quantity $1-e^{-pz^4}$ which occurs under the integral sign as a power series of pz^4, and evaluate the integral term by term. However, writing the Bessel function in the form

$$J_0(z)=\frac{1}{\pi}\mathrm{Re}\int_{-1}^{+1}e^{izt}\frac{dt}{\sqrt{1-t^2}}, \tag{A6}$$

and regarding z and t as complex variables, it is possible to choose integration paths along which this expansion will converge. Using the contours introduced in Sec. II C, the function $f(p)$ can be rewritten

$$f(p)=\frac{1}{\pi}\mathrm{Re}\int_\tau\frac{dt}{\sqrt{1-t^2}}\int_{\zeta_{\omega_t}}e^{izt}(1-e^{-pz^4})\frac{dz}{z^3}. \tag{A7}$$

We readily verify that the real parts of izt and $-pz^4$ are always negative, so the convergence of Eq. (A5) is not disturbed. With these new contours, it is now possible to expand the integrand in a power series of pz^4, and integrate term by term. For our purposes, it is only necessary to consider the term of first order in this expansion:

$$f(p)=p\frac{1}{\pi}\mathrm{Re}\int_\tau\frac{dt}{\sqrt{1-t^2}}\int_{\zeta_{\omega_t}}e^{izt}z\,dz+O(p^2). \tag{A8}$$

Therefore,

$$f'(0)=\frac{1}{\pi}\mathrm{Re}\int_\tau\frac{dt}{\sqrt{1-t^2}}\int_{\zeta_{\omega_t}}e^{izt}z\,dz. \tag{A9}$$

The integration on z can be carried out equivalently along the line ζ_{ψ_t}, defined in Sec. II C, on which $izt=-y$, $y\geqslant 0$ real. We obtain

$$f'(0)=-\frac{1}{\pi}\mathrm{Re}\int_{-1}^{+1}\frac{dt}{\sqrt{1-t^2}}\int_0^{+\infty}e^{-y}\frac{y}{t^2}dy, \tag{A10}$$

where t is a complex variable running along the semi-circle of radius unity lying in the domain $\mathrm{Im}(t)\geqslant 0$. Since

$$\int_{-1}^{+1}\frac{dt}{t^2\sqrt{1-t^2}}=0, \tag{A11}$$

we find $f'(0)=0$. Therefore

$$\frac{\partial F}{\partial(\sigma^2)}(\boldsymbol{\rho},\mathbf{0})=0 \quad \text{if} \quad \boldsymbol{\rho}\neq\mathbf{0}. \tag{A12}$$

To prove formula (93), it remains to show that

$$\int\frac{\partial F}{\partial(\sigma^2)}(\boldsymbol{\rho},\mathbf{0})d^2\boldsymbol{\rho}=\frac{\pi n}{4}\gamma^2\ln N. \tag{A13}$$

For this purpose, we introduce the function

$$I(\sigma^2)=\int F(\boldsymbol{\rho},\boldsymbol{\sigma})d^2\boldsymbol{\rho}. \tag{A14}$$

Substituting explicitly for $F(\boldsymbol{\rho},\boldsymbol{\sigma})$ and introducing polar coordinates, we obtain

$$I(\sigma^2)=\gamma^2\int_0^{+\infty}\rho\,d\rho\int_0^{+\infty}J_0(\rho\Phi)(1-e^{-\pi n\ln N\sigma^2\Phi^4/4})\frac{d\Phi}{\Phi^3}. \tag{A15}$$

Under this form, it is not possible to interchange the order of integration. An alternative expression for $I(\sigma^2)$ can be obtained along the following lines. Writing

$$I(\sigma^2)=\gamma^2\int_0^{+\infty}d\rho\int_0^{+\infty}\rho\Phi J_0(\rho\Phi)g(\Phi)d\Phi, \tag{A16}$$

where

$$g(\Phi)=(1-e^{-\pi n\ln N\sigma^2\Phi^4/4})\frac{1}{\Phi^4}, \tag{A17}$$

and integrating by parts with the identity

$$xJ_0(x)=\frac{d}{dx}(xJ_1(x)), \tag{A18}$$

we obtain

$$I(\sigma^2)=-\gamma^2\int_0^{+\infty}d\rho\int_0^{+\infty}J_1(\rho\Phi)\Phi g'(\Phi)d\Phi. \tag{A19}$$

It is now possible to interchange the order of integration. Since

$$\int_0^{+\infty}J_1(x)dx=1, \tag{A20}$$

Eq. (A19) reduces to

$$I(\sigma^2)=-\gamma^2\int_0^{+\infty}g'(\Phi)d\Phi=\gamma^2 g(0). \tag{A21}$$

Hence

$$I(\sigma^2)=\frac{\pi n}{4}\gamma^2\ln N\sigma^2. \tag{A22}$$

This formula is valid for any value of σ, but, for our purposes, we only need the result

$$I'(0)=\frac{\pi n}{4}\gamma^2\ln N. \tag{A23}$$

Since, by definition,

$$I'(0)=\int\frac{\partial F}{\partial(\sigma^2)}(\boldsymbol{\rho},\mathbf{0})d^2\boldsymbol{\rho}, \tag{A24}$$

we have proved Eq. (A13).

APPENDIX B: GENERALIZATION TO INCLUDE A SPECTRUM OF CIRCULATIONS AND AN ARBITRARY ISOTROPIC DISTRIBUTION OF VELOCITIES

So far, we have assumed that the system was a ''vortex plasma'' consisting of an equal number of vortices with circulation $+\gamma$ and $-\gamma$. We shall now indicate how the previous results can be extended to include a spectrum of circulations among the vortices. We shall also relax assumption (59) concerning the velocity distribution of the vortices, and generalize the results of Sec. III to any isotropic distribution $\tau(\mathbf{v})=\tau(|\mathbf{v}|)$ of the velocities. Such a distribution can be written conveniently in the form

$$\tau(\mathbf{v})=\int\frac{\tau(\mathbf{v}_0)}{2\pi v_0}\delta(v-v_0)d^2\mathbf{v}_0. \tag{B1}$$

If $\tau(\gamma)$ governs the distribution over the circulations, it is clear that Eq. (56) has to be modified according to

$$A_N(\boldsymbol{\rho},\boldsymbol{\sigma})=\left(\int_{\gamma=-\infty}^{+\infty}\int_{|\mathbf{r}|=0}^{R}\int_{|\mathbf{v}|=0}^{+\infty}\tau(\gamma)\tau(\mathbf{r})\tau(\mathbf{v})\times e^{i(\boldsymbol{\rho}\cdot\boldsymbol{\Phi}+\boldsymbol{\sigma}\cdot\boldsymbol{\psi})}d\gamma d^2\mathbf{r}d^2\mathbf{v}\right)^N. \tag{B2}$$

There is no *a priori* restriction on the function $\tau(\gamma)$, but we shall be particularly interested in the case where the system is ''neutral,'' i.e., $\overline{\gamma}=\int\tau(\gamma)\gamma d\gamma=0$. It is only in this circumstance that the velocity distribution (B1) may be used. Otherwise, there is a solid rotation of the system which adds to the dispersion of the particles.

Expression (66) for $A_N(\boldsymbol{\rho},\boldsymbol{\sigma})$ is now replaced by

$$A_N(\boldsymbol{\rho},\boldsymbol{\sigma})=\left(\frac{1}{\pi R^2}\int_{\gamma=-\infty}^{+\infty}\int_{|\mathbf{v}_0|=0}^{+\infty}\int_{|\boldsymbol{\Phi}|=|\gamma|/2\pi R}^{+\infty}\int_{|\boldsymbol{\psi}|=0}^{+\infty}\times\tau(\gamma)\frac{\tau(\mathbf{v}_0)}{2\pi v_0}\delta\left(\frac{|\gamma|\psi}{2\pi\Phi^2}-v_0\right)e^{i(\boldsymbol{\rho}\cdot\boldsymbol{\Phi}+\boldsymbol{\sigma}\cdot\boldsymbol{\psi})}\times\frac{\gamma^4}{16\pi^4\Phi^8}d\gamma d^2\mathbf{v}_0d^2\boldsymbol{\Phi}d^2\boldsymbol{\psi}\right)^N. \tag{B3}$$

Introducing polar coordinates, using identity (27) and substituting for

$$\delta\left(\frac{|\gamma|\psi}{2\pi\Phi^2}-v_0\right)=\frac{2\pi\Phi^2}{|\gamma|}\delta\left(\psi-v_0\frac{2\pi\Phi^2}{|\gamma|}\right) \tag{B4}$$

in Eq. (B3), we can easily integrate on ψ to obtain

$$A_N(\boldsymbol{\rho},\boldsymbol{\sigma})=\left(\frac{1}{\pi R^2}\int_{\gamma=-\infty}^{+\infty}\int_{|\mathbf{v}_0|=0}^{+\infty}\int_{|\boldsymbol{\Phi}|=|\gamma|/2\pi R}^{+\infty}\tau(\gamma)\tau(\mathbf{v}_0)\times e^{i\boldsymbol{\rho}\cdot\boldsymbol{\Phi}}J_0\left(\frac{2\pi\sigma}{|\gamma|}v_0\Phi^2\right)\frac{\gamma^2}{4\pi^2\Phi^4}d\gamma d^2\mathbf{v}_0d^2\boldsymbol{\Phi}\right)^N. \tag{B5}$$

It is readily verified that

$$\frac{1}{\pi R^2}\int_{\gamma=-\infty}^{+\infty}\int_{|\mathbf{v}_0|=0}^{+\infty}\int_{|\boldsymbol{\Phi}|=|\gamma|/2\pi R}^{+\infty}\tau(\gamma)\tau(\mathbf{v}_0)\frac{\gamma^2}{4\pi^2\Phi^4}\times d\gamma d^2\mathbf{v}_0d^2\boldsymbol{\Phi}=1. \tag{B6}$$

Therefore, the expression for $A_N(\boldsymbol{\rho},\boldsymbol{\sigma})$ can be rewritten equivalently,

$$A_N(\boldsymbol{\rho},\boldsymbol{\sigma})=\Bigg(1-\frac{1}{\pi R^2}\int_{\gamma=-\infty}^{+\infty}\int_{|\mathbf{v}|=0}^{+\infty}\int_{|\boldsymbol{\Phi}|=|\gamma|/2\pi R}^{+\infty}\tau(\gamma)\tau(\mathbf{v})\times\left[1-e^{i\boldsymbol{\rho}\cdot\boldsymbol{\Phi}}J_0\left(\frac{2\pi\sigma}{\gamma}v\Phi^2\right)\right]\frac{\gamma^2}{4\pi^2\Phi^4}\times d^2\boldsymbol{\Phi}d^2\mathbf{v}d\gamma\Bigg)^N, \tag{B7}$$

where we have written $\mathbf{v}$ instead of $\mathbf{v}_0$, as it is a dummy variable of integration. In the limit $N,R\to\infty$, with $n=N/\pi R^2$ fixed, we obtain

$$A(\boldsymbol{\rho},\boldsymbol{\sigma})=e^{-nC(\boldsymbol{\rho},\boldsymbol{\sigma})} \tag{B8}$$

with

$$C(\boldsymbol{\rho},\boldsymbol{\sigma})=\int_{\gamma=-\infty}^{+\infty}\int_{|\mathbf{v}|=0}^{+\infty}\int_{|\gamma|/2\pi R}^{+\infty}\tau(\gamma)\tau(\mathbf{v}_0)\times\left[1-J_0(\rho\Phi)J_0\left(\frac{2\pi\sigma}{|\gamma|}v\Phi^2\right)\right]\frac{\gamma^2}{2\pi\Phi^3}d\gamma d^2\mathbf{v}d\Phi. \tag{B9}$$

For $\boldsymbol{\sigma}=\mathbf{0}$, the function $C(\boldsymbol{\rho},\boldsymbol{\sigma})$ reduces to

$$C(\boldsymbol{\rho})\simeq\frac{\overline{\gamma^2\rho^2}}{16\pi}\ln N\quad(\rho>0), \tag{B10}$$

$$C(\boldsymbol{\rho})\sim-\frac{\overline{\gamma^2\rho^2}}{8\pi}\ln\rho\quad(\rho\to0). \tag{B11}$$

Therefore, the velocity distribution becomes

$$W(\mathbf{V})=\frac{4}{n\overline{\gamma^2}\ln N}e^{-(4\pi/n\overline{\gamma^2}\ln N)V^2}\quad[V\lesssim V_{crit}(N)], \tag{B12}$$

$$W(\mathbf{V})=\frac{n\overline{\gamma^2}}{4\pi^2V^4}\quad[V\gtrsim V_{crit}(N)], \tag{B13}$$

with

$$V_{crit}(N)\sim\left(\frac{n\overline{\gamma^2}}{4\pi}\ln N\right)^{1/2}\ln^{1/2}(\ln N). \tag{B14}$$

These results differ from Eqs. (34), (49), and (50) simply by the substitution $\gamma^2\to\overline{\gamma^2}$, where $\overline{\gamma^2}=\int\tau(\gamma)\gamma^2d\gamma$ is the average enstrophy.

Writing

$$C(\boldsymbol{\rho},\boldsymbol{\sigma})=C(\boldsymbol{\rho})+F(\boldsymbol{\rho},\boldsymbol{\sigma}), \tag{B15}$$

we obtain

$$F(\boldsymbol{\rho},\boldsymbol{\sigma})=\int_{\gamma=-\infty}^{+\infty}\int_{|\mathbf{v}|=0}^{+\infty}\int_{0}^{+\infty}\tau(\gamma)\tau(\mathbf{v})J_0(\rho\Phi)\times\left[1-J_0\left(\frac{2\pi\sigma}{|\gamma|}v\Phi^2\right)\right]\frac{\gamma^2}{2\pi\Phi^3}d\gamma d^2\mathbf{v}d\Phi. \tag{B16}$$

Following the same steps as in Appendix A, we find that

$$\frac{\partial F}{\partial(\sigma^2)}(\boldsymbol{\rho},\mathbf{0})=\pi^2\overline{v^2}\delta(\boldsymbol{\rho}), \tag{B17}$$

where

$$\overline{v^2}=\int_0^{+\infty}\tau(|\mathbf{v}|)v^2 2\pi v\,dv \tag{B18}$$

is the mean square velocity of the vortices for the isotropic distribution $\tau(|\mathbf{v}|)$. Therefore, Eqs. (95) and (96) are modified according to

$$\langle A^2\rangle_{\mathbf{V}}=\frac{n^2\overline{\gamma^2}}{4}\overline{v^2}\ln Ne^{(4\pi V^2/n\overline{\gamma^2}\ln N)}\quad[V\lesssim V_{crit}(N)], \tag{B19}$$

$$\langle A^2\rangle_{\mathbf{V}}=\frac{4\pi^2\overline{v^2}}{\overline{\gamma^2}}V^4\quad[V\gtrsim V_{crit}(N)], \tag{B20}$$

and Eqs. (103) and (104) according to

$$T(V)=\frac{2V}{n\sqrt{\overline{\gamma^2}}\sqrt{\ln N}\sqrt{\overline{v^2}}}e^{-(2\pi V^2/n\overline{\gamma^2}\ln N)}\quad[V\lesssim V_{crit}(N)] \tag{B21}$$

$$T(V)=\frac{\sqrt{\overline{\gamma^2}}}{2\pi\sqrt{\overline{v^2}}V}\quad[V\gtrsim V_{crit}(N)]. \tag{B22}$$

The mean duration of the fluctuations is

$$\langle T\rangle=\frac{2}{3\sqrt{6}}\frac{1}{\sqrt{n\overline{v^2}}}, \tag{B23}$$

and the diffusion coefficient is

$$D=\frac{1}{72}\left(\frac{6}{\pi}\right)^{1/2}\sqrt{\overline{\gamma^2}}\sqrt{\ln N}. \tag{B24}$$

For a Gaussian distribution of the velocities, we recover the results of Sec. III, appropriately generalized to account for a distribution over the circulations.

For $\boldsymbol{\rho}=\mathbf{0}$, the function $C(\boldsymbol{\rho},\boldsymbol{\sigma})$ reduces to

$$C(\boldsymbol{\sigma})=\int_{\gamma=-\infty}^{+\infty}\int_{|\mathbf{v}|=0}^{+\infty}\int_{0}^{+\infty}\tau(\gamma)\tau(\mathbf{v})\times\left[1-J_0\left(\frac{2\pi\sigma}{|\gamma|}v\Phi^2\right)\right]\frac{\gamma^2}{2\pi\Phi^3}d\gamma d^2\mathbf{v}d\Phi. \quad (B25)$$

Integrating by parts and using the identity

$$\int_0^{+\infty}\frac{J_1(x)}{x}dx=1, \quad (B26)$$

we obtain

$$C(\boldsymbol{\sigma})=\frac{1}{2}\overline{|\gamma|}\,\overline{v}\sigma, \quad (B27)$$

where

$$\overline{v}=\int_0^{+\infty}\tau(|\mathbf{v}|)v2\pi v\,dv \quad (B28)$$

is the average velocity of the vortices. Equation (78) is changed to

$$A(\boldsymbol{\sigma})=e^{-(n\overline{|\gamma|}/2)\overline{v}\sigma} \quad (B29)$$

and the Cauchy distribution (79) to

$$W(\mathbf{A})=\frac{2}{\pi n^2\overline{|\gamma|}^2\overline{v}^2}\frac{1}{\left(1+\frac{4A^2}{n^2\overline{|\gamma|}^2\overline{v}^2}\right)^{3/2}}. \quad (B30)$$

We also find

$$\langle V^2\rangle_{\mathbf{A}}=\frac{n\overline{\gamma^2}}{4\pi}\ln N \quad [A\lesssim A_{crit}(N)], \quad (B31)$$

$$\langle V^2\rangle_{\mathbf{A}}=\frac{\overline{\gamma^2}A}{2\pi\overline{|\gamma|}\,\overline{v}} \quad [A\gtrsim A_{crit}(N)], \quad (B32)$$

with

$$A_{crit}(N)\sim\frac{1}{2}\overline{|\gamma|}n\overline{v}\ln N. \quad (B33)$$

For a Gaussian distribution of the velocities, we recover the results of Sec. III, appropriately generalized to account for a distribution over the circulations.

APPENDIX C: CASE OF VORTEX BLOBS

In reality, the vortices have a finite radius a which is not necessarily small. This finiteness is responsible for a maximum allowable velocity $V_{max}\sim\gamma/4\pi a$, achieved when two vortices are at distance $\sim 2a$ from each other. Higher velocities are forbidden because the subsequent evolution is marked by merging events (see, e.g., Sire and Chavanis [16]). It is interesting to consider the distribution of velocity $\mathbf{V}$ and acceleration $\mathbf{A}$ produced by a collection of vortex ''blobs'' with finite radius. This problem was previously treated by Jiménez [11] and Min *et al.* [12] using numerical methods. The theory developed in this paper makes it possible to obtain new analytical results.

Introducing a cutoff at $r=a$, Eq. (29) is replaced by

$$C_N(\boldsymbol{\rho})=\frac{\gamma^2\rho^2}{2\pi}\int_{\gamma\rho/2\pi R}^{\gamma\rho/2\pi a}[1-J_0(x)]\frac{dx}{x^3}. \quad (C1)$$

In the limit $N,R\to\infty$ with $n=N/\pi R^2$ finite, we obtain

$$C(\boldsymbol{\rho})=\frac{\gamma^2\rho^2}{8\pi}\ln\left(\frac{R}{a}\right). \quad (C2)$$

In the case of vortex blobs, the singularity at $\boldsymbol{\rho}=\mathbf{0}$ is removed and the characteristic function $C_N(\boldsymbol{\rho})$ is exactly quadratic. Therefore $W(\mathbf{V})$ is the Gaussian equation (34) for all $V\leqslant V_{max}$. There is no algebraic tail in the limit considered. However, the convergence to the limit distribution (34) is still slow (see [11]) and, in practice, the velocity distribution can differ substantially from the Gaussian even for large values of N (note that the formalism presented in this paper could be used to study the dependence of the velocity distribution with the number N of vortices).

More interesting is the distribution of the acceleration $\mathbf{A}$. In the case of vortex blobs, Eq. (76) is replaced by

$$C(\boldsymbol{\sigma})=\frac{\gamma^2}{2\pi}\int_0^{\gamma/2\pi a}(1-e^{-\pi n\ln N\sigma^2\Phi^4/4})\frac{d\Phi}{\Phi^3}. \quad (C3)$$

After integrating by parts, one obtains

$$C(\boldsymbol{\sigma})=\pi a^2(e^{-(n\gamma^4\ln N/64\pi^3a^4)\sigma^2}-1)+\frac{\gamma^2}{8}\sqrt{n\ln N}\,\mathrm{Erf}\left(\frac{\gamma^2}{8\pi^2a^2}\sqrt{n\pi\ln N}\sigma\right)\sigma, \quad (C4)$$

where

$$\mathrm{Erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-y^2}dy \quad (C5)$$

is the ''error function.''

For $\sigma\to 0$, the function $C(\boldsymbol{\sigma})$ is quadratic in σ,

$$C(\boldsymbol{\sigma})\sim\frac{n\gamma^4\ln N\sigma^2}{64\pi^2a^2} \quad (|\boldsymbol{\sigma}|\to 0), \quad (C6)$$

implying that the distribution $W(\mathbf{A})$ is Gaussian for large values of the acceleration:

$$W(\mathbf{A})\sim\frac{16\pi a^2}{n^2\gamma^4\ln N}e^{-(16\pi^2a^2/n^2\gamma^4\ln N)A^2} \quad (|\mathbf{A}|\to+\infty). \quad (C7)$$

Its variance is

$$\langle A^2\rangle=\frac{n^2\gamma^4}{16\pi^2a^2}\ln N. \quad (C8)$$

For $\sigma \rightarrow +\infty$, the function $C(\boldsymbol{\sigma})$ is linear in σ,

$$C(\boldsymbol{\sigma}) \sim \frac{\gamma^2}{8}\sqrt{n \ln N}\,\sigma \quad (|\boldsymbol{\sigma}| \rightarrow +\infty), \tag{C9}$$

and we recover the Cauchy distribution (79) for small values of $|\mathbf{A}|$. Therefore, the distribution $W(\mathbf{A})$ makes a smooth transition from Cauchy (concave on a semilog plot) for small fluctuations to Gaussian (convex on a semilog plot) for large fluctuations. It is likely that in between the distribution passes through an *exponential tail*, as observed numerically by Min *et al.* [12] for the velocity gradients. Of course, when a is reduced, the transition between the two regimes occurs at larger fluctuations [see Eq. (C8)]. According to Eqs. (81) and (C8) the relevant nondimensional parameter to consider is the *area fraction* na^2. In decaying turbulence, the influence of an extended core should be manifest at the beginning of the evolution, when na^2 is large.

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